Paper No.
Scalable Hybrid Preconditioners

F. Nataf, Laboratory J.L. Lions
P. Jolivet (PhD student)
LINEAR SOLVERS

Domain decomposition methods
Multigrid methods

Projection methods
Relaxation-based methods

“Hybrid” methods

Multifrontal factorizations
Supernodal factorizations

Iterative methods

Direct methods

Multigrid:

++ N log(N) complexity
++ Algebraic Multigrid

-- Based on heuristics that need to be tuned and are not available for all kinds of systems

Paper # • Scalable Hybrid Preconditioners • F. Nataf
DOMAIN DECOMPOSITION METHODS

Naturally parallel framework

Consider the linear system: $Au = f \in \mathbb{R}^n$. Given a decomposition of $[1; n]$, $(\mathcal{N}_1, \mathcal{N}_2)$, define:

- the restriction operator $R_i$ from $[1; n]$ into $\mathcal{N}_i$,
- $R_j^T$ as the extension by 0 from $\mathcal{N}_i$ into $[1; n]$.

Then solve concurrently:

$$u_1^{m+1} = u_1^m + A_{11}^{-1} R_1 (f - Au^m) \quad u_2^{m+1} = u_2^m + A_{22}^{-1} R_2 (f - Au^m)$$

where $u_i = R_i u$ and $A_{ij} := R_i AR_j^T$. 
DOMAIN DECOMPOSITION METHODS

We have effectively divided, but we have yet to conquer. *Duplicated* unknowns coupled via a *partition of unity*:

\[ I = \sum_{i=1}^{N} R_i^T D_i R_i. \]

Then, \( u^{m+1} = \sum_{i=1}^{N} R_i^T D_i u_i^{m+1} \).

\[ M_{RAS}^{-1} = \sum_{i=1}^{N} R_i^T D_i A_{ii}^{-1} R_i. \]

Requires an adequate coarse space correction a.k.a two level methods.
TWO LEVEL PRECONDITIONERS

A common technique in the field of MG, DDM and deflation:

introduce an auxiliary “coarse” problem.

Let $Z$ be a rectangular matrix. Define

$$ E := Z^T A Z. $$

$Z$ has $O(N)$ columns, hence $E$ is much smaller than $A$. Enrich the original preconditioner, e.g. additively

$$ P^{-1} = M^{-1} + Z E^{-1} Z^T, $$

Cf [Tang, Nabben, Vuik, Erlangga]
ADAPTIVE COARSE SPACE Z

- Generalized Eigenvalue in the Overlap problem *per subdomain*

\[ A_{j}p_{j,k} = \lambda_{j,k} X_{j}A_{j}^{o}X_{j} p_{j,k} \quad (X_{j} \ldots \text{diagonal}) \]

Choose first \( m_j \) eigenvectors per subdomain

\[ Z := R_{j}^{T} D_{j} p_{j,k}^{k=1,...,m_j \ j=1,...,N} \]

GENEO automatically includes Near Null Kernel a.k.a Zero Energy Modes

- Comparison with existing works:
  Nataf et al. (SIAM 2011), Galvis, Efendiev, Lazarov & Willems (2011)
ADAPTIVE COARSE SPACE Z

Theorem (Spillane, Dolean, Hauret, Nataf, Pechstein, Scheichl (Num. Math. 2013)):

\[
\kappa(M_{ASM,2}^{-1}A) \leq (1 + k_0)[2 + k_0(2k_0 + 1) \max_{j=1}^N \left(1 + \frac{1}{\lambda_{j,m_j+1}}\right)]
\]

Possible criterion for picking \( m_j \):
\( H_j \)… subdomain diameter, \( \delta_j \) … overlap

\( \lambda_{j,m_j+1} < \frac{\delta_j}{H_j} \)

Or fixed size \( m_j \)
FIRST NUMERICAL RESULTS

Darcy

Channels and inclusions: $1 < K < 1.5 \times 10^6$
FIRST NUMERICAL RESULTS

Optimality

$m_i$ is given automatically by the strategy.

<table>
<thead>
<tr>
<th>#Z per subd.</th>
<th>ASM</th>
<th>ASM+$Z_{ Nico}$</th>
<th>ASM+$Z_{ Geneo}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max(m_i - 1, 1)$</td>
<td></td>
<td></td>
<td>273</td>
</tr>
<tr>
<td>$m_i$</td>
<td>614</td>
<td>543</td>
<td>36</td>
</tr>
<tr>
<td>$m_i + 1$</td>
<td></td>
<td></td>
<td>32</td>
</tr>
</tbody>
</table>

Taking one fewer eigenvalues has a huge influence on the iteration count.

Taking one more has only a small influence.
NUMERICAL PLATFORM

Library interfaced with FreeFem++, Feel++

 Runs on laptops (Linux, OSX, Windows), clusters, HPC machines

Curie Thin Nodes
5,040 compute nodes.
2 eight-core Intel Sandy Bridge@2.7 GHz per node.
1.7 PFLOPs peak performance.
BullxMPI based on an old OpenMPI (1.6.x).
3 million core hours through two PRACE calls.
STRONG SCALING (linear elasticity)

Coefficients

Solution

Metis or Scotch decomposition

Paper # • • Scalable Hybrid Preconditioners • F. Nataf
STRONG SCALING (linear elasticity)

1 subdomain / MPI process, 2 OpenMP threads/MPI process
STRONG SCALING (linear elasticity)

1 subdomain / MPI process, 2 OpenMP threads/MPI process

2.1B d.o.f. in 2D ($\mathbb{P}_3$ FE)
300M d.o.f. in 3D ($\mathbb{P}_2$ FE)

<table>
<thead>
<tr>
<th>#processes</th>
<th>1024</th>
<th>2048</th>
<th>4096</th>
<th>8192</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio</td>
<td>100%</td>
<td>80%</td>
<td>60%</td>
<td>40%</td>
</tr>
</tbody>
</table>

Factorization  Deflation vectors  Coarse operator  Krylov method
WEAK SCALING (Darcy)

\[ \nabla \cdot (\kappa \nabla u) = f \]

+ BC
WEAK SCALING (Darcy)

1 subdomain / MPI process, 2 OpenMP threads/MPI process
WEAK SCALING (Darcy)

1 subdomain / MPI process, 2 OpenMP threads/MPI process
BENCHMARKING WITH AMG

Comparison performance of setup and solution phases between our solver against purely algebraic (+ near null space) solvers:

Hypre BoomerAMG – algebraic multigrid (LNLL)
  Solution strategy optimized with the help of the developers

GAMG - algebraic multigrid (ANL/LBL)
  Solution strategy optimized with the help of the developers:
  a relative threshold for dropping edges in the aggregation graphs of 0.2
  a six-dimensional near-null space set to the rigid body modes—three translations and three rotations.

Tests were made via a PETSc interface to FreeFem++. 
NUMERICAL RESULTS

Homogeneous 3D linear elasticity equation discretized by P2 FE solved on 4,096 MPI processes, 262M d.o.f.

![Graph showing relative residual error and time comparison between different methods: Schwarz GenEO, Hypre BoomerAMG, and PETSc GAMG. The x-axis represents the number of iterations, and the y-axis represents the relative residual error. The chart indicates performance and convergence of the methods over iterations. Additionally, a bar chart displays the time taken for setup and solution, comparing Schwarz GenEO and BoomerAMG. The results highlight the efficiency of the hybrid preconditioners in terms of both accuracy and computational time.](image-url)
NUMERICAL RESULTS

Heterogeneous 3D linear elasticity equation discretized by P2 FE solved on 4,096 MPI processes, 262M d.o.f.
FINAL WORDS

Summary:
scalable framework for building two-level preconditioners for both
Schwarz (shown here) or substructuring methods (FETI-1, BDD)
possible interface (FEM, FVM) without a global ordering

Limitations:
scaling of the coarse operator in 3D beyond 10k subdomains, cf. Falgout and Schroder 2014]
deflation vectors need elementary matrices to be computed

Prospects:
recycling of the coarse space for non linear problems
saddle point problems
Acknowledgements / Thank You / Questions

PRACE  (Partnership for Advanced Computing in Europe)

FreeFem++ (Open Source Free package) developed by F. Hecht